Math 601 − **Spring** 2014
Homework #4

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1. (a) Given handlebody decompositions of surfaces S and S', each admitting a single $0-$ and $2-$handle, determine a handlebody decomposition of the connect sum $S\sharp S'$, again admitting a single $0-$ and $2-$handle.
   (b) Show that every closed, orientable surface is diffeomorphic to S^2 or $\sharp_k T^2$, where T^2 is the torus.
   (c) Show that every closed, unorientable surface can be decomposed as $\mathbb{RP}^2\sharp S'$.
   (d) Prove that $T^2\sharp\mathbb{RP}^2$ is diffeomorphic to $\mathbb{RP}^2\sharp\mathbb{RP}^2\sharp\mathbb{RP}^2$.
   (e) List all closed surfaces.
2. Prove that every finitely presented group is the fundamental group of some closed, orientable smooth 4-manifold (fun fact: $\mathbb{Z}\rtimes Z$ does not arise as the fundamental group of any 3-manifold).
3. Let Y_g be the double of the genus g handlebody. Determine $\pi_1(Y_g)$ and $H_k(Y_g)$. What is the manifold Y_g?
4. Prove that $Hom(-,G)$ is left-exact. Conclude that $Ext^0(H,G)\cong Hom(H,G)$.
5. (Hatcher, number 2.2.21) If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X)=\chi(A)+\chi(B)-\chi(A\cap B)$.
6. (Hatcher, number 2.2.22) For X a finite CW complex and $p:\tilde{X}\to X$ an n-sheeted covering space, show that $\chi(\tilde{X})=n\chi(X)$.
7. (Hatcher, number 3.1.3) Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free $\mathbb{Z}_4$-modules. Use this to show that $Ext^n_{\mathbb{Z}_4}(\mathbb{Z}_2,\mathbb{Z}_2)$ is nonzero for all n.
8. (Hatcher, number 3.1.10) For the Lens space $L_m(l_1,\cdots,l_n)$ defined in Example 2.43, compute the cohomology groups using the cellular cochain complex and taking coefficients in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_m$, and $\mathbb{Z}_p$ for p prime. Verify that the answers agree with those given by the universsal coefficient theorem.